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A RECURRENCE PROPERTY OF SMOOTH FUNCTIONS

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ABSTRACT

Let P and Q be real polynomials of degrees d and e, respectively, and f a periodic function. It is shown that, if f is s times differentiable at $Q(0)$, where $s \geqq 7de^3 \log 14e^3$, then for every $\varepsilon > 0$ the diophantine inequality

$$|P(x)f(Q(x)) - P(0)f(Q(0)) - y| < \varepsilon, \qquad x \neq 0,$$

has a solution. This settles in particular a question raised by Furstenberg and Weiss [6].

1. Recurrence properties of polynomials

Let $\| x \|$ denote the distance from x to the nearest integer. The well-known Weyl's equidistribution theorem [8] implies that, given a real polynomial P and $\varepsilon > 0$, there exists a non-trivial solution to the diophantine inequality

$$\| P(x) - P(0) \| < \varepsilon. \tag{1.1}$$

Obviously, this implies that (1.1) actually has infinitely many solutions. Moreover, the set of solutions is "large" in the sense that it intersects any IP-set. We shall not elaborate here on the notion of an IP-set (see [5] for an extensive treatment of IP-sets), but only mention that the foregoing means in particular that the set of solutions of (1.1) is *relatively dense*, namely it is of bounded gaps [5, Lemma 9.2].

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2. Recurrence for smooth functions

It turns out that products of functions recurring often arbitrarily close to their values at 0, i.e. satisfying an inequality such as (1.1) for "many" (in the sense discussed in the preceding section) values of x, need not be such. Furstenberg and Weiss [6, Sec. 6] showed this by proving that, whereas the functions x and $\cos x\alpha$ (for any fixed α) obviously have the recurrence property in question, the function $x \cos x\alpha$ is not such for a.e. α. This was accomplished as follows. Le Veque [7] proved that, for a.e. α, the sequence $(x \cos x\alpha)_{x=1}^{\infty}$ is uniformly distributed modulo 1. Furstenberg and Weiss proved that, if $(x_k)_{k=1}^{\infty}$ is a sufficiently rapidly growing sequence of integers, then for any positive integer d the sequence

$$(x_k \cos x_k\alpha, (x_k+1)\cos(x_k+1)\alpha, \ldots, (x_k+d-1)\cos(x_k+d-1)\alpha)$$

is uniformly distributed modulo $(1, 1, \ldots, 1)$ for a.e. α. Now for every such α there exist in particular arbitrarily large blocks of consecutive integers such that, for each x belonging to one these blocks, we have, say, $\| x \cos x\alpha \| > \frac{1}{4}$; in other words, the set of solutions of $\| x \cos x\alpha \| < \frac{1}{4}$ cannot be relatively dense.

In view of the above, it was asked in [6] whether for every α and $\varepsilon > 0$ the inequality $\| x \cos x\alpha \| < \varepsilon$ has a non-trivial solution.

We shall show that the answer to this question is affirmative. In fact, we shall obtain the rather more general

THEOREM 2.1. *Let $P, Q \in \mathbf{R}[x]$ be polynomials of degrees d and e, respectively, and $f: \mathbf{R} \to \mathbf{R}$ a periodic function, s times differentiable at* 0, *where*

$$s \geqq 7de^3 \log 14e^3. \tag{2.1}$$

Then for every $\varepsilon > 0$ there exists a positive integer x such that

$$\| P(x)f(Q(x)) - P(0)f(Q(0)) \| < \varepsilon. \tag{2.2}$$

In Section 3 we discuss various improvements of the theorem for special cases. The proof itself is presented in Section 4.

3. Improving Theorem 2.1

The proof of Theorem 2.1, to be carried out in the next section, makes use of the following result of R. C. Baker.

THEOREM A [2, Th. 2]. *Let $1 \leqq a_1 < a_2 < \cdots < a_k \leqq e$ be integers. There* *exists a constant $D = D(e)$ having the following property. For any real numbers $c_1, c_2, \ldots, c_k$ and positive integer N there exists an integer $1 \leqq x \leqq N$ with*

$$\| c_j x^{a_j} \| < DN^{-1/(7ke^2 \log 14ke^2)}, \qquad 1 \leqq j \leqq k. \tag{3.1}$$

This theorem will be applied as follows. Suppose the polynomial Q of Theorem 2.1 is given by $Q(x) = \sum_{j=0}^{e} c_j x^j$. By Theorem A, the system of diophantine inequalities

$$\| c_j x^j \| < Dx^{-1/(7e^3 \log 14e^3)}, \qquad 1 \leqq j \leqq e, \tag{3.2}$$

has infinitely many solutions. The right-hand side of (2.1) is just the product of the degree of P and the denominator of the exponent of the right-hand side of (3.2). Going over the proof of Theorem 2.1 (see Section 4), it is easily verified that, if the right-hand side of (3.2) can be replaced by $Cx^{-1/r}$, then the right-hand side of (2.1) can be replaced by dr. Thus, any future improvement of Theorem A will automatically yield a corresponding improvement of Theorem 2.1.

For some special types of polynomials, improved versions of Theorem A are available. If some of the coefficients of Q vanish, then Theorem A itself leads to a strengthening of Theorem 2.1. For example, if $Q(x) = c_0 + c_e x^e$, then the right-hand side of (2.1) reduces to $7de^2 \log 14e^2$. Moreover, in this case Theorem A can be improved as follows for $e \geqq 9$ [4]: For every sufficiently large N there exists some $1 \leqq x \leqq N$ with

$$\| c_e x^e \| < N^{-\log e / 4e(\log e + 1)\log(e \log e + 1)}.$$

Hence, if $Q(x) = c_0 + c_e x^e$, $e \geqq 9$, then (2.1) may be replaced by

$$s \leqq 4de(\log e + 1)\log(e \log e + 1)/\log e.$$

Results improving upon Theorem A are also known if $e \leqq 3$, with corresponding improvements of Theorem 2.1. In fact, for $e = 1$, already the classical Dirichlet's theorem allows us to replace the right-hand side of (3.1) by N^{-1}, so we only need $s \geqq d$ in (2.1). Similarly, for $e = 2$, the right-hand side of (3.1) can be replaced by $N^{-1/4+\varepsilon}$ for arbitrarily small $\varepsilon > 0$ [3], and thus we have $s \geqq 4d + 1$ instead of (2.1). For $e = 3$, the right-hand side of (3.1) can be reduced to $N^{-1/12+\varepsilon}$ [2], giving $s \geqq 12d + 1$ in place of (2.1).

4. Proof of Theorem 2.1

Let $P(x) = b_0 + b_1x + \cdots + b_dx^d$ and $Q(x) = c_0 + c_1x + \cdots c_ex^e$. In view of Theorem A there exists an ascending sequence $(x_k)_{k=1}^{\infty}$ of positive integers such that for each k we have

$$\| c_j x_k^j \| < D x_k^{-1/7e^3 \log 14e^3}, \qquad 1 \leqq j \leqq e,$$

for an appropriately chosen constant D. Without loss of generality, we may assume f to be of period 1. For a real number t, denote by $\{t\}$ the number in $(-\frac{1}{2}, \frac{1}{2}]$ for which $t - \{t\}$ is an integer. Define a polynomial Q_1 by $Q_1(x) = Q(x) - c_0$. Let h be an arbitrary fixed positive integer. Then

$$P(hx_k)f(Q(hx_k)) = P(hx_k) \cdot f(c_0 + \{Q_1(hx_k)\})$$

$$= \sum_{i=0}^{d} b_i(hx_k)^i \cdot \left(\sum_{j=0}^{s} \frac{f^{(j)}(c_0)}{j!} \{Q_1(hx_k)\}^j + o(\{Q_1(hx_k)\}^s) \right).$$

Now

$$\left| \sum_{i=0}^{d} b_i(hx_k)^i \cdot o(\{Q_1(hx_k)\}^s) \right| \leqq Bx_k^d \cdot o\left(\left(\sum_{j=1}^{e} h^j \| c_j x_k^j \| \right)^s \right)$$

$$\leqq Bx_k^d h^{es} e^s D^s \cdot o(x_k^{-s/7e^3 \log 14e^3}) = B_1(h) \cdot o(x_k^{-\eta})$$

for certain constants B, $B_1(h)$ and $\eta \geqq 0$. For all sufficiently large k we have

$$\{Q_1(hx_k)\} = \sum_{j=1}^{e} h^j \{c_j x_k^j\}.$$

Consequently

$$P(hx_k)f(Q(hx_k)) = R(h) + o(1),$$

where R is a polynomial of degree not exceeding $d + es$ in h whose free term is $b_0 f(c_0)$ and whose other coefficients are polynomials in x_k, $\{c_1x_k\}, \{c_2x_k^2\}, \ldots, \{c_ex_k^e\}$. Replacing (x_k) by a suitably chosen subsequence thereof, we may assume all these coefficients to converge modulo 1 as $k \to \infty$, and therefore

$$P(hx_k)f(Q(hx_k)) \xrightarrow[k \to \infty]{} P(0)f(Q(0)) + \sum_{j=1}^{d+es} \alpha_j h^j \pmod 1$$

for certain $\alpha_1, \alpha_2, \ldots, \alpha_{d+es}$. Weyl's theorem now completes the proof.

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